Solutions of state space equations (LTI systems)

Preliminaries

$$e^{\boxed{A}t} = \underline{I} + \underline{A}t + \frac{\underline{A}^2 t^2}{2!} + \frac{\underline{A}^3 t^3}{3!} + \dots$$

matrix

$$= \sum_{i=0}^{\infty} \frac{(\underline{A}t)^i}{i!}$$

Properties

$$e^{(\underline{A}+\underline{B})t} = e^{\underline{A}t} \cdot e^{\underline{B}t}$$

$$\left(e^{\underline{A}t}\right)^T = e^{\underline{A}^T t}$$

$$\frac{d}{dt}\left[e^{\underline{A}t}\right] = \underline{A}e^{\underline{A}t} = e^{\underline{A}t}\underline{A}$$

$$\frac{d}{dt}\left[e^{-\underline{A}(t-t_0)}\underline{x}(t)\right] = -\underline{A}e^{-\underline{A}(t-t_0)}\underline{x}(t) + e^{-\underline{A}(t-t_0)}\dot{\underline{x}}(t)$$

$$= -e^{-\underline{A}(t-t_0)}\underline{A}\underline{x}(t) + e^{-\underline{A}(t-t_0)}\dot{\underline{x}}(t)$$

$$= e^{-\underline{A}(t-t_0)}\left[\dot{\underline{x}}(t) - \underline{A}\underline{x}(t)\right]$$

Solutions to SS equations

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$\underline{x(t_0)}$

given $u(t)$, and $x(t_0)$, what is $x(t)$
once $x(t)$ is found we can find $y(t)$.

$$\dot{x} = Ax + Bu$$

$$\dot{x} - Ax = Bu$$

$$e^{-\underline{A}(\tau - t_0)}\left[\dot{x} - Ax\right] = e^{-\underline{A}(\tau - t_0)} Bu.$$

$$\frac{d}{d\tau}\left[e^{-\underline{A}(\tau - t_0)} x(\tau)\right] = e^{-\underline{A}(\tau - t_0)} Bu.$$

$$d\left[e^{-\underline{A}(\tau - t_0)} x(\tau)\right] = e^{-\underline{A}(\tau - t_0)} Bu\,d\tau$$

$$\int_{t_0}^{t} d\left[e^{-\underline{A}(\tau - t_0)} x(\tau)\right] = \int_{t_0}^{t} e^{-\underline{A}(\tau - t_0)} Bu\,d\tau$$

$$\underbrace{-e^{-\underline{A}\cdot 0} x(t_0)}_{x(t_0)} + e^{-\underline{A}(t - t_0)} x(t) = \int_{t_0}^{t} e^{-\underline{A}(\tau - t_0)} Bu\,d\tau$$

$$e^{-\underline{A}(t - t_0)} x(t) = x(t_0) + \int_{t_0}^{t} e^{-A(\tau - t_0)} Bu\,d\tau$$

$$x(t) = e^{\underline{A}(t - t_0)} x(t_0) + e^{\underline{A}(t - t_0)} \int_{t_0}^{t} e^{-A(\tau - t_0)} Bu\,d\tau$$

$$\boxed{x(t) = e^{\underline{A}(t - t_0)} x(t_0) + \int_{t_0}^{t} e^{-A(\tau - t)} Bu\,d\tau}$$

$$\boxed{\phi(t) = e^{At}}$$

$$\boxed{x(t) = \phi(t - t_0)\, x(t_0) + \int_{t_0}^{t} \phi(t - \tau)\, Bu(\tau)\,d\tau}$$

if $t_0 = 0$ (initial condition are zero)

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

if $B=0$ and $u(t)$ is a step or constant.
if A is invertable.

$$\int_0^t e^{A(t-\tau)} B \, d\tau = A^{-1}\left(e^{At} - I\right)B$$

COMPUTING e^{At}

General method.

$$e^{At} = \mathcal{L}^{-1}\left[(SI-A)^{-1}\right]$$

EX:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

$$X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

u: unit step input.

$$SI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[sI - A\right] = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} - 2e^{-2t} \end{bmatrix}$$

$$= e^{At}$$

$$\int_0^t e^{A(t-t_0)} B\,d\tau = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\,d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} - 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$y = x_1(t) =$

Method 2

If a matrix A can be transformed into a diagonal canonical form then,

$$e^{At} = Pe^{Dt}P^{-1}$$

where P is the transformation matrix such that

$$P^{-1}AP = D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$d_1 \ldots d_n$ are the eigen values of A.

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & & \\ & \ddots & \\ & & e^{d_n t} \end{bmatrix}$$

Proof

let $\dot{x} = Ax$

$$x(t) = e^{At}\, x(0) \qquad (1)$$

let $x = Pz$ ← diagonalising matrix

$$z = P^{-1}x$$

$$\dot{z} = P^{-1}\dot{x} = P^{-1}Ax$$

$$\dot{z} = \underbrace{P^{-1}AP}_{D}\, z = Dz$$

$$z(t) = e^{Dt}\, z(0)$$

$$P^{-1}x = e^{Dt}\, P^{-1}x(0)$$

$$x(t) = \underbrace{Pe^{Dt}P^{-1}}_{e^{At}}\, x(0) \qquad (2)$$

$$\boxed{e^{At} = Pe^{Dt}P^{-1}}$$

If A can be transformed into Jordon canonical form then

$$e^{At} = \bar{P}e^{Jt}\bar{P}^{-1}$$

where $\bar{P}$ is a transformation matrix such that

$$\bar{P}^{-1}A\bar{P} = J$$ ← Jordon canonical form

Ex:

$$J = \left[\begin{array}{cc|ccc} \lambda_1 & 1 & & & \\ 0 & \lambda_1 & & & \\ \hline & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{array}\right]$$

λ_1 is repeated 2 times.

$$e^{Jt} = \left[\begin{array}{cc|ccc} e^{\lambda_1 t} & te^{\lambda_1 t} & & & \\ 0 & e^{\lambda_1 t} & & & \\ \hline & & e^{\lambda_3 t} & & \\ & & & \ddots & \\ & & & & e^{\lambda_n t} \end{array}\right]$$

Ex:

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} \end{bmatrix}$$